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NARROW-BAND SYSTEMS AND GAUSSIANTITY

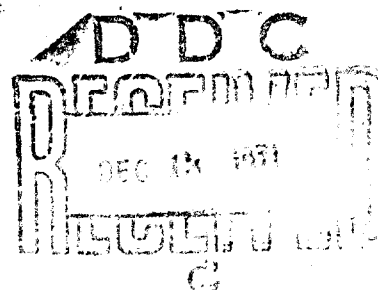
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NARROW-BAND SYSTEMS AND GAUSSIANTITY

A. Papoulis

Polytechnic Institute of Brooklyn

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FOREWORD

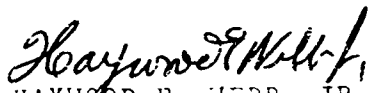
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ABSTRACT

The approach to Gaussianity of the output $y(t)$ of a narrow-band system $h(t)$ is investigated. It is assumed that the input $x(t)$ is an a-dependent process in the sense that the random variables $x(t)$ and $x(t+u)$ are independent for $u > a$. With $F(y)$ and $G(y)$ the distribution function of $y(t)$ and of a suitable normal process, a realistic bound B of the difference $F(y) - G(y)$ is determined and it is shown that $B \rightarrow 0$ as the bandwidth ω_0 of the system tends to zero. In the special case of the shot noise process

$$y(t) = \sum_i h(t-t_i)$$

it is shown that

$$|F(y) - G(y)| < \sqrt{\frac{\omega_0}{\lambda}}$$

where λ is the average density of the Poisson points t_i .

1. Introduction

INTRODUCTION

In the engineering applications of random signals it is often assumed that if a stationary process $x(t)$ is the input to a linear system, then the resulting response

$$y(t) = \int_{-\infty}^{\infty} x(t-\alpha) h(\alpha) d\alpha \quad (1)$$

tends to a normal process as the bandwidth ω_0 of the system tends to zero.

This theorem is not always true as one can show with a trivial counter example. However, it holds under fairly general conditions. To apply it meaningfully, we need to establish not only conditions for its asymptotic validity, but also realistic bounds for the deviation of $\underline{y}(t)$ from Gaussianity for a given $\omega_0 \neq 0$.

As one might expect from the central limit theorem, $\underline{y}(t)$ will approach Gaussianity if the past $\underline{x}(u)$, $u \leq t$ of the process $\underline{x}(t)$ is "almost" independent of its future $\underline{x}(u)$, $u \geq t + \tau$ for sufficiently large τ . This loose requirement is precisely formulated in Rosenblatt's classic paper [1] as follows:

Let B_t and F_τ be the Borel fields generated by the random variables $\underline{x}(u)$ for $u \leq t$ and $u \geq t + \tau$ respectively. We say that the process $\underline{x}(t)$ satisfies the strong mixing condition if there is a function $g(\alpha)$ with

$$0 \leq g(\alpha) \downarrow 0 \quad \text{as} \quad \alpha \rightarrow \infty \quad (2)$$

such that for any pair of events $B \in B_t$, $F \in F_\tau$

$$|P(BF) - P(B)P(F)| < g(\tau) \quad (3)$$

Assuming further that the moments of $\underline{x}(t)$ of order up to four exist and satisfy certain conditions and that its power spectrum $S(\omega)$ is such that

$$S(\omega) > 0 \quad (4)$$

he shows that the output $\underline{y}(t)$ of a certain class of filters tends to Gaussianity.

In applying Rosenblatt's result, we are faced with the problem of

testing the mixing condition (3). Furthermore, the problem of establishing realistic bounds for the distance of $\underline{y}(t)$ from Gaussianity remains. In this paper we shall overcome these difficulties but only at the sacrifice of generality. We shall base our analysis on the assumption that the process $\underline{x}(t)$ is a -dependent, i. e., that the events B and F are independent for $\tau > a$. This assumption is equivalent to the condition

$$g(\alpha) = 0 \quad \text{for} \quad \alpha > a \quad (5)$$

The above is, of course, more restrictive than (2), however, it holds in many applications and can be often readily established. Suppose, for example, that $\underline{x}(t)$ is the output of a memory-less non-linear system with input a normal process $\underline{s}(t)$. In this case, condition (5) is equivalent to the assumption that the autocovariance $C(\tau)$ of $\underline{s}(t)$ vanishes for $\tau > a$.

2. The Berry-Esseen Theorem

The output $\underline{y}(t)$ of our system is a linear combination of a non-countable infinity of dependent random variables. As we show in section 5, if the input $\underline{x}(t)$ is a -dependent, then the principal part of $\underline{y}(t)$ can be expressed as a sum of independent random variables. To solve our problem we shall need, therefore, a bound of the deviation of such a sum from Gaussianity. Such a bound is given in the following important theorem due to A. C. Berry and G. Esseen [2, 3]. We present it here for easy reference and also because in the proof we make use of a useful lemma (Appendix A) which is an improvement of the corresponding lemma 1 in Feller [2, p. 510]. With its help, we obtain the constant 4 in (8), whereas, the corresponding constant in Feller is $33/4$. However, as it is stated in [2], unpublished calculations mention the constant 2.9 (Esseen, 1956) and 2.05 (Wallace, 1958).

Consider a sequence $\underline{x}_1, \underline{x}_2, \dots$ of independent r. v. (random

variables) such that

$$E\{\underline{x}_i\} = 0, \quad E\{\underline{x}_i^2\} = \sigma_i^2, \quad E\{|\underline{x}_i|^3\} = c_i$$

We form the sum

$$\underline{x} = \frac{1}{\sigma} \sum_i \underline{x}_i \quad \text{where} \quad \sigma^2 = \sum_i \sigma_i^2 < \infty \quad (6)$$

Clearly, $E\{\underline{x}^2\} = 1$. With $F(x)$ the distribution function of \underline{x} and

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy \quad (7)$$

that of a normal r. v. with the same mean and variance, we wish to bound the "distance" $|F(x) - G(x)|$ of \underline{x} from Gaussianity.

Theorem. If

$$\frac{c_i}{\sigma_i^2} \leq \lambda$$

then

$$|F(x) - G(x)| < 4 \frac{\lambda}{\sigma} \quad (8)$$

Proof. With $\Phi_i(w)$ and $\Phi(w)$ the characteristic functions of the r. v. \underline{x}_i and \underline{x} respectively, we conclude from (6) and the independence of the \underline{x}_i 's that

$$\Phi(w) = \Phi_1\left(\frac{w}{\sigma}\right) \cdot \Phi_2\left(\frac{w}{\sigma}\right) \cdots \quad (9)$$

As we see from (B-6)

$$\Phi_i(w) = e^{-\frac{\sigma_i^2 w^2}{2} + \theta_i \frac{5c_i}{24} |w|^3}, \quad |w| \leq \frac{1}{2\lambda} \leq \frac{\sigma_i^2}{2c_i} \quad (10)$$

(throughout the paper the letter θ will always be such that $|\theta| \leq 1$),

hence,

$$\Phi(\omega) = e^{-\frac{\omega^2}{2} + \theta \frac{5}{21} \frac{|\omega|^3}{\sigma}} \sum_i c_i = e^{-\frac{\omega^2}{2} + \theta \frac{5\lambda}{21\sigma} |\omega|^3}, \quad |\omega| \leq \frac{\sigma}{2\lambda} \quad (11)$$

because

$$\frac{\sum_i c_i}{\sum_i \sigma_i^2} < \max_i \frac{c_i}{\sigma_i^2} \leq \lambda$$

But for any z

$$\left| \frac{e^z - 1}{z} \right| = \left| 1 + \frac{z}{2} + \frac{z^2}{3!} + \dots \right| \leq 1 + |z| + \frac{|z|^2}{2} + \dots = e^{|z|} \quad (12)$$

hence,

$$\left| \Phi(\omega) - e^{-\frac{\omega^2}{2}} \right| \leq e^{-\frac{\omega^2}{2}} \left| e^{\frac{5\theta\lambda}{21\sigma} |\omega|^3} - 1 \right| \leq$$

$$|\omega| \leq \frac{\sigma}{2\lambda}$$

$$\frac{5\lambda}{21\sigma} |\omega|^3 e^{-\frac{\omega^2}{2}} \left[1 - \frac{10\lambda}{21\sigma} |\omega| \right] < \frac{5\lambda}{21\sigma} |\omega|^3 e^{-\frac{\omega^2}{4}} \quad (13)$$

We now introduce a function $r(x)$ as in (A-1) with Fourier transform $R(\omega)$ such that

$$R(\omega) = 0 \quad \text{for} \quad |\omega| > \omega_1 \equiv \frac{\sigma}{2\lambda} \quad (14)$$

As we see from (A-1)

$$|R(\omega)| \leq R(0) = 1 \quad (15)$$

The transform of the convolution

$$g(x) \equiv [F(x) - G(x)] * r(x) \quad (16)$$

is given by [4]

$$\frac{\hat{\Phi}(\omega) - e^{-\omega^2/2}}{j\omega} R(\omega)$$

hence,

$$2\pi |g(x)| = \left| \int_{-\omega_1}^{\omega_1} \frac{\hat{\Phi}(\omega) - e^{-\omega^2/2}}{\omega} R(\omega) d\omega \right| \leq \frac{5\lambda}{21\sigma} \int_{-\omega_1}^{\omega_1} \omega^2 e^{-\omega^2/4} d\omega$$

$$< \frac{5\lambda}{21\sigma} \int_{-\infty}^{\infty} \omega^2 e^{-\omega^2/4} d\omega = \frac{5\lambda}{21\sigma} 4\sqrt{\pi}$$

Thus

$$|g(x)| \leq \frac{10\lambda}{21\sigma\sqrt{\pi}} \quad (17)$$

Since

$$G'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \leq \frac{1}{\sqrt{2\pi}}$$

we conclude from (17) and (A-15) that

$$|F(x) - G(x)| \leq \gamma \sqrt{\frac{2}{\pi}} \quad (18)$$

where the constant γ is such that

$$\int_0^{\gamma} r(x) dx = \frac{1}{3} + \frac{5\sqrt{2}\lambda}{63\sigma\gamma} \quad (19)$$

We now choose for $r(x)$ the Fejer kernel [4]

$$r(x) = \frac{\sin^2(\omega_1 x/2)}{\pi \omega_1 x^2/2} \quad (20)$$

whose transform is a triangle satisfying (14). And with

$$w = \frac{\omega_1 \gamma}{2} = \frac{\sigma \gamma}{4 \lambda} \quad (21)$$

we conclude that

$$|F(x) - G(x)| \leq \frac{4\lambda}{\sigma} \sqrt{\frac{2}{\pi}} w \quad (22)$$

where w is such that

$$\frac{1}{\pi} \int_0^w \frac{\sin^2 x}{x^2} dx = \frac{1}{3} + \frac{5}{126\sqrt{2} w} \quad (23)$$

Solving we find

$$w \approx 1.26$$

Inserting into (22) we obtain (8).

Corollary. If the r. v. \underline{z}_i are independent with

$$E\{\underline{z}_i\} = 0, \quad E\{\underline{z}_i^2\} = \sigma^2, \quad E\{|\underline{z}_i|^3\} \leq c \quad (24)$$

and

$$\underline{z} = \sum_i \alpha_i \underline{z}_i, \quad |\alpha_i| \leq A, \quad \sum \alpha_i^2 = \alpha^2$$

then

$$\left| F(z) - G\left(\frac{z}{\alpha\sigma}\right) \right| \leq 4 \frac{Ac}{\alpha\sigma^3} \quad (25)$$

The above follows from (8) with $\underline{x}_i = \alpha_i \underline{z}_i$.

3. Systems driven by an impulse train

Consider a band-limited system with energy E

$$E = \int_{-\infty}^{\infty} h^2(t) dt \quad H(\omega) = 0 \quad \text{for} \quad |\omega| \geq \omega_0 \quad (26)$$

As it is known [5],

$$|h(t)| \leq \sqrt{\frac{E\omega_0}{\pi}}, \quad \sum_{n=-\infty}^{\infty} h^2(t-nT) = \frac{E}{T} \quad \text{for } T < \frac{\pi}{\omega_0} \quad (27)$$

With \tilde{z}_n a sequence of r.v. as in (24) we form the impulse train

$$\tilde{x}(t) = \sum_m \tilde{z}_m \delta(t-nT) \quad T < \frac{\pi}{\omega_0} \quad (28)$$

If $\tilde{x}(t)$ is the input to our system, then the resulting output is given by

$$y(t) = \sum_n \tilde{z}_n h(t-nT) \quad (29)$$

Clearly,

$$F\{\tilde{y}(t)=0\}, \quad E\{\tilde{y}^2(t)\} \equiv \sigma_y^2 = \frac{E\sigma^2}{T}$$

With $F(y)$ the distribution function of $y(t)$, it follows from (25) and (27) that

$$F(y) - G\left(\frac{y}{\sigma_y}\right) \leq \frac{4c}{\sigma^3} \sqrt{\frac{T\omega_0}{\pi}} \quad (30)$$

From the above it follows that $\tilde{y}(t)$ tends to Gaussianity with $\omega_0 \rightarrow 0$.

4. Shot noise

Consider the random points \tilde{t}_n of a Poisson process with average density λ . If the sequence of impulses

$$\tilde{x}(t) = \sum_n \delta(t-\tilde{t}_n) \quad (31)$$

is the input to our system, then the resulting output

$$\underline{y}(t) = \sum_n h(t-t_{\underline{n}}) \quad (32)$$

is the familiar shot noise process [6, 7]. As it is known, the characteristic function $\Phi_o(\omega)$ of $\underline{y}(t)$ is given by [8, p. 567]

$$\Phi_o(\omega) = e^{-\lambda \int_{-\infty}^{\infty} [e^{j\omega h(t)} - 1] dt} \quad (33)$$

and its mean η and variance σ^2 by (Campbell's theorem [8, p. 569])

$$\eta = \lambda \int_{-\infty}^{\infty} h(t) dt, \quad \sigma^2 = \lambda \int_{-\infty}^{\infty} h^2(t) dt = \lambda E \quad (34)$$

To simplify notations we shall consider the normalized process

$$\underline{s}(t) = \frac{\underline{y}(t) - \eta}{\sigma} \quad (35)$$

whose mean is zero, variance one, and characteristic function

$$\Phi(\omega) = e^{-j\frac{\eta\omega}{\sigma}} \Phi_o\left(\frac{\omega}{\sigma}\right) \quad (36)$$

With $F(s)$ the distribution function of $\underline{s}(t)$ we shall show that

$$|F(s) - G(s)| < \frac{2.6}{\pi} \sqrt{\frac{\omega_o}{\lambda}} \quad (37)$$

Proof. Since [see (B-6)]

$$e^{j\omega h(t)} = 1 - j\omega h(t) - \frac{\omega^2 h^2(t)}{2} + \theta(t) \frac{\omega^3 h^3(t)}{6}, \quad |\theta(t)| \leq 1 \quad (38)$$

we conclude from (33) and (36) that all ω

$$\Phi(\omega) = e^{-\frac{\omega^2}{2} + \theta \beta \omega^3} \quad |\theta| \leq 1 \quad (39)$$

where

$$\beta \equiv \frac{\lambda \int_{-\infty}^{\infty} |h(t)|^3 dt}{6\sqrt{\lambda^3 E^3}} \leq \frac{1}{6} \sqrt{\frac{\omega_0}{\pi \lambda}} \quad (40)$$

because [see (27)]

$$\int_{-\infty}^{\infty} |h(t)|^3 dt < \sqrt{\frac{E\omega_0}{\pi}} \int_{-\infty}^{\infty} h^2(t) dt = E \sqrt{\frac{E\omega_0}{\pi}} \quad (41)$$

From (39) and (12) we obtain

$$\left| \Phi(\omega) - e^{-\omega^2/2} \right| = e^{-\frac{\omega^2}{2}} \left| e^{\beta \omega^3} - 1 \right| \leq \beta |\omega|^3 e^{-\frac{\omega^2}{2} + \beta |\omega|^3} \quad (42)$$

valid for all ω . With $\epsilon < 1$ a constant to be soon determined, we observe that

$$-\frac{\omega^2}{2} + \beta |\omega|^3 < -\frac{\omega^2}{2\epsilon^2} \quad \text{for} \quad |\omega| \leq \frac{1}{2\beta} \left(1 - \frac{1}{\epsilon^2}\right) \equiv \omega_1 \quad (43)$$

hence,

$$\left| \Phi(\omega) - e^{-\omega^2/2} \right| \leq \beta |\omega|^3 e^{\frac{\omega^2}{2\epsilon^2}}, \quad |\omega| \leq \omega_1 \quad (44)$$

We now proceed as in section 2: with $r(x)$ and $g(x)$ as in (16), we have

$$2\pi |g(x)| = \left| \int_{-\omega_1}^{\omega_1} \frac{\Phi(\omega) - e^{-\omega^2/2}}{\omega} R(\omega) d\omega \right| \leq \beta \int_{-\omega_1}^{\omega_1} \omega^2 e^{-\omega^2/2} \epsilon^2 d\omega < \beta \epsilon^2 \sqrt{2\pi}$$

Therefore, [see (17) to (23)]

$$|g(x)| \leq \frac{8\epsilon^3}{\sqrt{2\pi}} \quad (45)$$

and

$$|F(x) - G(x)| \leq \frac{2w}{w_1} \sqrt{\frac{2}{\pi}} = \frac{8\theta}{\sqrt{2\pi}} \frac{\epsilon^2}{\epsilon^2 - 1} w \quad (46)$$

where w is such that

$$\frac{1}{\pi} \int_0^w \frac{\sin^2 x}{x^2} dx = \frac{1}{3} + \frac{\epsilon(\epsilon^2 - 1)}{24w} \quad (47)$$

It remains to solve (47) for w as a function of ϵ , and find ϵ such as to minimize (46). We find that the optimum ϵ and the corresponding w are given by

$$\epsilon^2 \simeq 3 \quad w \simeq 1.9$$

Inserting into (46) we obtain (37). The bound (44) yields

$$\left| \Phi(w) - e^{-w^2/2} \right| \leq \theta |w|^3 e^{-w^2/6}, \quad |w| < \frac{1}{3\theta}$$

The value of ϵ is not critical.

5. Band-limited systems with a -dependent inputs

Consider a stationary process with zero mean, autocorrelation $R(\tau)$, and power spectrum $S(w)$

$$E\{\underline{x}(t)\} = 0, \quad R(\tau) = E\{\underline{x}(t+\tau)\underline{x}(t)\} \longleftrightarrow S(w) \quad (48)$$

We shall assume that $\underline{x}(t)$ is a -dependent [10] as defined in section 1.

From the definition it follows that if the instances t_r and t_s are such

that

$$\max_r t_r + a < \min_s t_s \quad (49)$$

then the r.v. $\tilde{x}(t_r)$ and $\tilde{x}(t_s)$ are independent. Hence,

$$R(\tau) = 0 \quad \text{for} \quad |\tau| > a, \quad S(0) = \int_{-a}^a R(\tau) d\tau \quad (50)$$

We define the constant α by

$$\frac{1}{a} \int_{-a}^a |\tau| R(\tau) d\tau = \alpha S(0) \quad (51)$$

It is easy to see that $\alpha < 1$.

In this section we shall bound the deviation of the output $\tilde{y}(t)$ of our system (26) in terms of the above quantities and the third absolute moment

$$E \{ |\tilde{x}(t)|^3 \} = c \quad (52)$$

of the input $\tilde{x}(t)$.

For this purpose, we express the impulse response $h(t)$ of our system as a sum (Fig. 1)

$$h(t) = \bar{h}(t) + \epsilon(t) \quad (53)$$

of a staircase function

$$\bar{h}(t) = h(nT) \quad (n - \frac{1}{2})T \leq t < (n + \frac{1}{2})T \quad (54)$$

and an error term $\epsilon(t)$, where T is such that

$$2a < T < \frac{\pi}{\omega_0} \quad T \equiv a + b \quad (55)$$

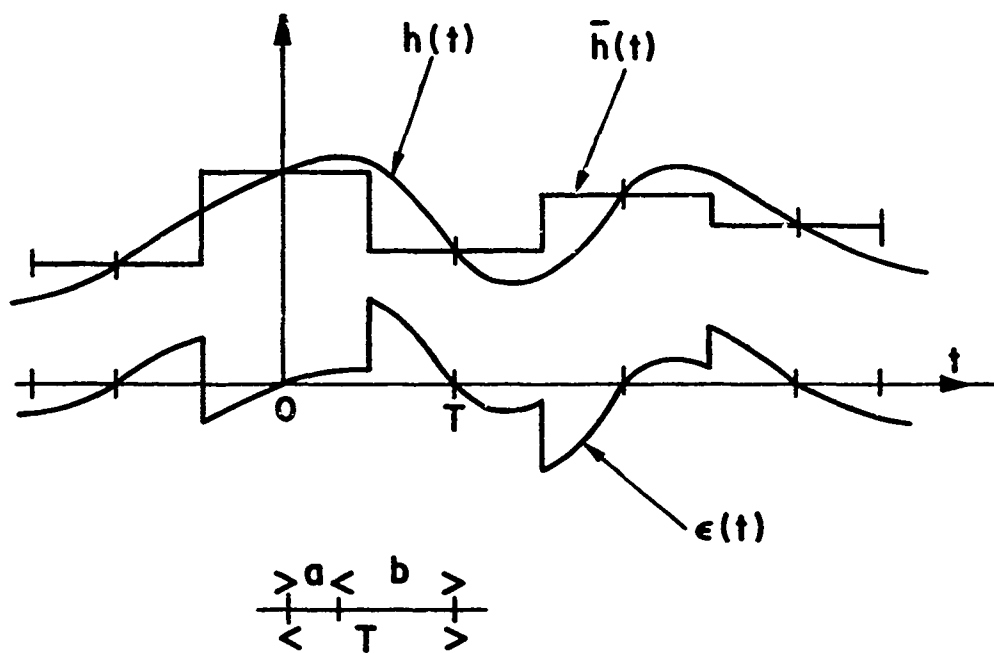


Fig. 1

We next form the r. v.

$$\begin{aligned} z_n &= \int_{(n-\frac{1}{2})T}^{(n+\frac{1}{2})T} x(t-\alpha) d\alpha \\ w_n &= \int_{(n-\frac{1}{2})T+b}^{(n+\frac{1}{2})T} x(t-\alpha) d\alpha \end{aligned} \quad (56)$$

and

$$\begin{aligned} z &= \sum_n h(nT) z_n \\ w &= \sum_n h(nT) w_n \end{aligned} \quad (57)$$

As we see from (53), the output $y(t)$ of the system is given by

$$y(t) = \int_{-\infty}^{\infty} \bar{h}(\alpha) x(t-\alpha) d\alpha + e \quad (58)$$

where

$$e = \int_{-\infty}^{\infty} \epsilon(\alpha) x(t-\alpha) d\alpha \quad (59)$$

From the above it follows that

$$y(t) = z + w + e \quad (60)$$

As we shall presently see, if

$$a \ll T \ll \pi/\omega_0 \quad (61)$$

then the dominant term in (60) is the r. v. z .

From (56) and (57) and the stationarity of $x(t)$ it follows that

[9 p. 346]

$$E \{ \underline{z}_n \} = 0, \quad E \{ \underline{w}_n \} = 0$$

$$E \{ \underline{z}_n^2 \} = b \int_{-a}^a R(\tau) \left[1 - \frac{|\tau|}{b} \right] d\tau = S(0) (1 - \alpha \frac{a}{b}) b \quad (62)$$

$$E \{ \underline{w}_n^2 \} = a \int_{-a}^a R(\tau) \left[1 - \frac{|\tau|}{a} \right] d\tau = S(0) (1 - \alpha) a \quad (63)$$

$$E \{ |\underline{z}_n|^3 \} \leq \int_0^b \int_0^b \int_0^b E \{ |\underline{x}(\alpha_1) \underline{x}(\alpha_2) \underline{x}(\alpha_3)| \} d\alpha_1 d\alpha_2 d\alpha_3 \quad (64)$$

But

$$E^3 \{ |\underline{x} \underline{y} \underline{z}| \} \leq E \{ |\underline{x}|^3 \} E \{ |\underline{y}|^3 \} E \{ |\underline{z}|^3 \} \quad (65)$$

as one can show from Hölder's inequality, hence, [see (52)]

$$E \{ |\underline{z}_n|^3 \} \leq b^3 c \quad (66)$$

From the a -dependence of $\underline{x}(t)$ it follows that the r. v. \underline{z}_n are independent, hence, [see (27)]

$$\sigma_o^2 = E \{ \underline{z}^2 \} = \sum_n h^2(nT) E \{ \underline{z}_n^2 \} = \frac{E}{T} S(0) (1 - \alpha \frac{a}{b}) b \quad (67)$$

But $|h(nT)| \leq \sqrt{\frac{E w_o}{\pi}}$, therefore, [see (25) and (68)]

$$\left| F(z) - G\left(\frac{z}{\sigma_z}\right) \right| \leq A_1 T^2 \sqrt{w_o} \quad (68)$$

where

$$A_1 = \frac{4c^3}{\sqrt{\pi S^3(0) (1 - |\alpha| a/b)^3}} \quad (69)$$

The r. v. \underline{w}_n are also independent, hence,

$$\sigma_w^2 = E\{\underline{w}^2\} = \sum_n h^2(nT) E\{\underline{w}_n^2\} = E A_2^2 \frac{a}{T} \quad (70)$$

where

$$A_2^2 = S(0) (1-\alpha) \quad (71)$$

From (59) it follows that [9, p. 346]]

$$\sigma_e^2 < \rho(0) \int_{-a}^a R|\alpha| d\alpha < E_e 2aR(0) \leq E A_3^2 \frac{a}{T} \quad (72)$$

where

$$A_3^2 = \frac{1}{6} R(0) T^3 \omega_o^2 \quad (73)$$

From the above it follows that

$$E\{(\underline{w} + \underline{e})^2\} \leq (\sigma_w + \sigma_e)^2 = E(A_2 + A_3)^2 \frac{a}{T} \quad (74)$$

The constant T is still to be determined. If it is small, then the bound in (68) is small, however, the variance of the term $\underline{w} + \underline{e}$ is large. To account for its effect on the distribution $F_y(y)$ of $\underline{y}(t)$ [see (62)] we shall use the bound (D-3) in Appendix D. As we see from (74) and (67)

$$\frac{E \{ (\tilde{w} + \tilde{e})^2 \}}{E \{ \tilde{z}^2 \}} \leq \frac{E (A_2 + A_3)^2 \frac{a}{T}}{E S(0)(1 - \alpha a/b)} = A_4 \frac{a}{T} \quad (75)$$

where

$$A_4 = \frac{(A_2 + A_3)^2 (b+a)}{S_0 (b - \alpha a)} \quad (76)$$

We note that the quantities A_1 , A_2 and A_4 are essentially independent of the system. Furthermore, the final result is useful only if ω_0 is so small that $a \ll T \ll \pi/\omega_0$. In this case, $A_3 \ll A_2$, $b \simeq T$ and A_4 is close to unity.

From (75), (68), (60), and (D-4) it follows that

$$\left| F_y(y) - G\left(\frac{y}{\sigma_0}\right) \right| < A_1 T^2 \sqrt{\omega_0} + 1.03 \sqrt[3]{A_4 \frac{a}{T}} \quad (77)$$

Choosing T so as to minimize the above, we finally obtain

$$\left| F_y(y) - G\left(\frac{y}{\sigma_0}\right) \right| < 2 \sqrt[7]{A_1 A_4^2 a^2 \sqrt{\omega_0}} \quad (78)$$

From the preceding discussion it follows that if the input to a band-limited system is an a -dependent process such that

$$E \{ |\tilde{x}|^3 \} < \infty \text{ and } S(0) \neq 0$$

then the resulting output tends to Gaussianity with $\omega_0 \rightarrow 0$.

Appendix A

A useful lemma. Given an even non-negative function $r(x)$ of unit area

$$\int_{-\infty}^{\infty} r(x) dx = 1, \quad r(-x) = r(x), \quad r(x) \geq 0 \quad (A-1)$$

and a differentiable function $\varphi(x)$ such that

$$\varphi(\pm \infty) = 0, \quad \varphi'(x) \geq -A < 0 \quad (A-2)$$

we form the convolution integral

$$g(x) = \int_{-\infty}^{\infty} \varphi(\xi) r(x-\xi) d\xi = \varphi(x) * r(x) \quad (A-3)$$

We maintain that if

$$|g(x)| \leq B \quad (A-4)$$

then [9, 11, 12]

$$|\varphi(x)| \leq 2A\gamma \quad (A-5)$$

where γ is such that

$$\int_0^{\gamma} r(x) dx = \frac{1}{3} + \frac{B}{6A\gamma} \quad (A-6)$$

Proof. Suppose that the maximum of $\varphi(x)$ equals C . Without loss of generality we can assume that this maximum is reached at $x = 0$. Thus

$$|\varphi(x)| \leq |\varphi(0)| = C \quad (A-7)$$

Consider first the case $\varphi(0) > 0$. Clearly,

$$\varphi(x) - C = \int_0^x \varphi'(\xi) d\xi$$

Hence,

$$\varphi(x) \geq C - Ax \quad (A-8)$$

Furthermore,

$$g(x) = \int_{-\infty}^0 + \int_0^{2x} + \int_{2x}^{\infty} \varphi(\xi) r(x-\xi) d\xi$$

From (A-8) and the evenness of $r(x)$ it follows with $x - \xi = y$ that

$$\int_0^{2x} \varphi(\xi) r(x-\xi) d\xi \geq \int_0^{2x} (C - A\xi) r(x-\xi) d\xi = 2 [C - Ax] \int_0^x r(y) dy$$

But

$$\int_{-\infty}^0 + \int_{2x}^{\infty} \varphi(\xi) r(x-\xi) d\xi \leq C \left[\int_{-\infty}^0 + \int_{2x}^{\infty} r(x-\xi) d\xi \right] = C \left[1 - 2 \int_0^x r(y) dy \right]$$

Hence,

$$g(x) > [4C - 2Ax] \int_0^x r(y) dy - C$$

Letting

$$x = \frac{C}{2A} \quad (A-9)$$

in (A-9) we conclude with (A-4) that

$$B \geq 3C \int_0^{C/2A} r(y) dy - C \quad (A-10)$$

If our assertion (A-5) is not true then

$$\varphi(0) = C > 2A\gamma \quad (A-11)$$

But this leads to a contradiction because then (A-10) would yield [see (A-6)]

$$B \geq 3 C \int_0^{\gamma} r(y) dy = \frac{CB}{2A\gamma} > B$$

hence, (A-4) is true.

The case $\varphi(0) < 0$ can be reduced to the above. Indeed, with

$$\varphi_1(x) = -\varphi(-x), \quad g_1(x) = \varphi_1(x) * r(x)$$

we have

$$\varphi_1'(x) = \varphi'(-x) \quad g_1(x) = g(-x)$$

hence,

$$|\varphi_1(x)| \leq \varphi_1(0) > 0, \quad |g_1(x)| \leq B$$

From the above it follows that $|\varphi_1(x)| \leq 2A\gamma$ and the proof of the theorem is thus complete.

Corollary. Consider two distribution functions $F_1(x)$, $F_2(x)$ and a function $r(x)$ satisfying (A-1). With

$$g(x) = [F_1(x) - F_2(x)] * r(x) \tag{A-12}$$

we maintain that if

$$F_2'(x) \leq A \tag{A-13}$$

and

$$|g(x)| \leq B \tag{A-14}$$

then

$$|F_1(x) - F_2(x)| \leq 2A\gamma \tag{A-15}$$

where γ is as in (A-6).

Proof. With

$$\varphi(x) = F_1(x) - F_2(x)$$

we have $\varphi(\pm\infty) = 0$. Furthermore,

$$\varphi(x) \geq C - Ax$$

as we see from (A-13) and the monotonicity of $F_1(x)$. The desired bound (A-15) follows as in the lemma.

Appendix B

Consider a random variable \underline{x} such that

$$E\{\underline{x}\} = 0, \quad E\{\underline{x}^2\} = \sigma^2, \quad E\{|\underline{x}|^3\} = c \quad (\text{B-1})$$

With $f(x)$ its density function and

$$\Phi(\omega) = \int_{-\infty}^{\infty} f(x) e^{j\omega x} dx \quad (\text{B-2})$$

the corresponding characteristic function, we maintain that for all ω

$$|\Phi(\omega) - 1| \leq \frac{\omega^2 \sigma^2}{2} \quad (\text{B-3})$$

$$\left| \Phi(\omega) - 1 - \frac{\omega^2 \sigma^2}{2} \right| \leq \frac{\omega^3 c}{6} \quad (\text{B-4})$$

and

$$\Phi(\omega) = e^{-\frac{\sigma^2 \omega^2}{2} + \theta \frac{5c}{24} |\omega|^3} \quad \text{for } |\omega| < \frac{\sigma^2}{2c} \quad (\text{B-5})$$

Proof

As it is known [2, 13]

$$e^{j\omega x} = 1 + j\omega x + \theta_1 \frac{\omega^2 x^2}{2}, \quad |\theta_1| \leq 1 \quad (\text{B-6})$$

$$e^{j\omega x} = 1 + j\omega x - \frac{\omega^2 x^2}{2} + \theta_2 \frac{\omega^3 x^3}{6}, \quad |\theta_2| \leq 1$$

Inserting into (B-2) we obtain (B-3) and (B-4).

From the expansion

$$-\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \dots \quad |z| < 1 \quad (\text{B-7})$$

it follows with $z = 1 - \Phi(\omega)$ that if $|1 - \Phi(\omega)| < 1$, then

$$-\log\{1 - [1 - \Phi(\omega)]\} = [1 - \Phi(\omega)] + \sum_{n=2}^{\infty} \frac{1}{n} [1 - \Phi(\omega)]^n \quad (\text{B-8})$$

hence,

$$\left| \log \Phi(\omega) + \frac{1}{2} \omega^2 \sigma^2 \right| \leq \left| \Phi(\omega) - 1 - \frac{\omega^2 \sigma^2}{2} \right| + \sum_{n=2}^{\infty} \frac{1}{n} \left| 1 - \Phi(\omega) \right|^n \quad (\text{B-9})$$

As it is known, $\sigma^3 \leq c$; hence, for

$$|\omega| < \frac{\sigma^2}{2c} \leq \frac{1}{2\sigma} \quad (\text{B-10})$$

$$|1 - \Phi(\omega)| \leq \frac{\omega^2 \sigma^2}{2} \equiv r < \frac{1}{8}$$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{n} |1 - \Phi(\omega)|^n \leq \frac{r^2}{2} \left[1 + \frac{2}{3}r + \frac{2}{4}r^2 + \dots \right] < \frac{r^2}{2} \frac{1}{1-r} < \frac{4r^2}{7}$$

and [see (B-6), (B-9) and (B-4)]

$$\left| \log \Phi(\omega) + \frac{\omega^2 \sigma^2}{2} \right| \leq \frac{c|\omega|^3}{6} + \frac{\sigma^4 \omega^4}{7} < \frac{5c}{21} |\omega|^3 \quad (\text{B-11})$$

from which (B-5) follows.

Appendix C

In the decomposition

$$h(t) = \bar{h}(t) + \epsilon(t) \quad (\text{C-1})$$

of Fig. 1, the transform $\bar{H}(\omega)$ of $\bar{h}(t)$ is given by [14]

$$\bar{H}(\omega) = \frac{2 \sin(\omega T/2)}{T \omega} \sum_n H(\omega + \frac{n\pi}{T}) \quad (\text{C-2})$$

(Fig. 2) and its energy [15, p. 122] by

$$\bar{E} = \int_{-\infty}^{\infty} \bar{h}^2(t) dt = T \sum_n h^2(nT) = E \quad (\text{C-3})$$

We shall show that

$$E_{\epsilon} = \int_{-\infty}^{\infty} \epsilon^2(t) dt < \frac{E}{12} T^2 \omega_o^2 \quad (\text{C-4})$$

Proof. Since $H(\omega) = 0$ for $|\omega| \geq \omega_o$ we conclude from Parseval's formula that

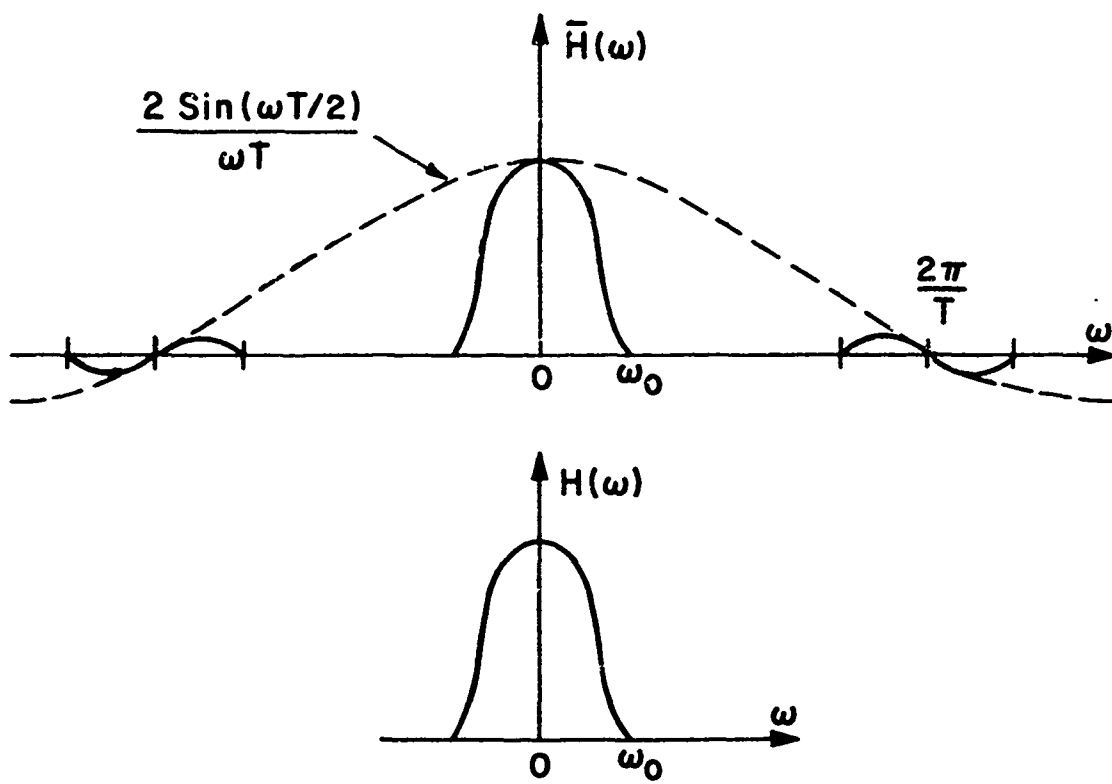


Fig. 2

$$\int_{-\infty}^{\infty} h(t) \bar{h}(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \bar{H}^*(\omega) d\omega = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} |H(\omega)|^2 \frac{2 \sin(\omega T/2)}{T \omega} d\omega \quad (C-5)$$

hence,

$$E_{\epsilon} = \int_{-\infty}^{\infty} [h(t) - \bar{h}(t)]^2 dt = \frac{1}{\pi} \int_{\omega_0}^{\omega_0} |H(\omega)|^2 \left[1 - \frac{2 \sin(\omega T/2)}{\omega T} \right] d\omega$$

But

$$|\sin x - x| \leq \frac{x^3}{6}$$

hence

$$E_{\epsilon} \leq \frac{T^2}{24\pi} \int_{-\omega_0}^{\omega_0} \omega^2 |H(\omega)|^2 d\omega \leq \frac{E}{12} T^2 \omega_0^2$$

Appendix D

The r.v. \underline{x} is such that $E\{\underline{x}\} = 0$, $E\{\underline{x}^2\} = 1$. With $F(x)$ its distribution function, we assume that [see (7)]

$$|F(x) - G(x)| < \delta \quad \text{for all } x \quad (D-1)$$

The r.v. \underline{y} is such that

$$E\{\underline{y}\} = 0 \quad E\{|\underline{y}|\} = M_1 \quad E\{\underline{y}^2\} = \sigma^2 \quad (D-2)$$

We form the sum

$$\underline{z} = \underline{x} + \underline{y}$$

With $F_z(z)$ the distribution function of \underline{z} we shall show that:

a) If the r.v. \underline{x} and \underline{y} are independent, then

$$|F_z(z) - G(z)| < \delta + \frac{\sigma}{\sqrt{2\pi}} \quad (D-3)$$

b) In any case,

$$|F_z(z) - G(z)| < \delta + 1.03 \sqrt[3]{\sigma^2} \quad (D-4)$$

Proof. a) With $f_y(y)$ the density of \underline{y} , it follows from the independence of \underline{x} and \underline{y} that

$$F_z(z) = \int_{-\infty}^{\infty} F(z-\xi) f_y(\xi) d\xi \quad (D-5)$$

hence,

$$F_z(z) - G(z) = \int_{-\infty}^{\infty} [F(z-\xi) - G(z-\xi) + G(z-\xi) - G(z)] f_y(\xi) d\xi \quad (D-6)$$

But

$$|G(z-\xi) - G(z)| = |G'(z-\theta\xi) \xi| \leq \frac{|\xi|}{\sqrt{2\pi}} \quad (D-7)$$

therefore,

$$|F_z(z) - G(z)| < \delta + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\xi| f_y(\xi) d\xi = \delta + \frac{M_1}{\sqrt{2\pi}}$$

and (D-3) follows because $M_1 < \sigma$.

b) $F_z(z)$ equals the probability masses in the region $x + y \leq z$ of the x - y plane. With ϵ a constant to be soon determined, we see that

$$F(z-\epsilon) - P\{\underline{y} > \epsilon\} < F_z(z) < F(z+\epsilon) + P\{\underline{y} < -\epsilon\} \quad (D-8)$$

hence, [see (D-1)]

$$G(z-\epsilon) - \delta - P\{y > \epsilon\} - G(z) < F_z(z) - G(z) < G(z+\epsilon) + \delta + P\{y < -\epsilon\} - G(z)$$

and [see (D-7)]

$$|F_z(z) - G(z)| < \delta + \frac{\epsilon}{\sqrt{2\pi}} + P\{|y| > \epsilon\} \quad (D-9)$$

It remains to select ϵ so as to minimize the above bound. From Tchebycheff's inequality we have [9, p. 150]

$$P\{|y| \geq \epsilon\} \leq \frac{\sigma^2}{\epsilon^2}$$

hence,

$$|F_z(z) - G(z)| < \delta + \frac{\epsilon}{\sqrt{2\pi}} + \frac{\sigma^2}{\epsilon^2} \quad (D-10)$$

This quantity is minimum for

$$\epsilon = \sqrt[3]{2\sigma^2\sqrt{2\pi}}$$

Inserting into (D-10) we obtain (D-4).

If the variance of \tilde{x} is not one but σ_x^2 , then

$$\left|F(x) - G\left(\frac{x}{\sigma_x}\right)\right| < \delta + \sqrt[3]{\frac{27\sigma^2}{8\pi\sigma_x^2}} \quad (D-11)$$

This follows readily from (D-4) by a simple scaling.

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